

LOCAL MULTIFRACTAL ANALYSIS IN METRIC SPACES

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ABSTRACT. We study the local dimensions and local multifractal properties of measures on doubling metric spaces. Our aim is twofold. On one hand, we show that there are plenty of multifractal type measures in all metric spaces which satisfy only mild regularity conditions. On the other hand, we consider a local spectrum that can be used to gain finer information on the local behaviour of measures than its global counterpart.

1. INTRODUCTION

In multifractal analysis, the interest is in the behaviour of the local dimension map

$$x \mapsto \dim_{\text{loc}}(\mu, x) = \lim_{r \downarrow 0} \log \mu(B(x, r)) / \log r,$$

for some, often dynamically defined, fractal type measures μ . From the mathematical point of view, the ultimate goal is to understand the size of the level sets

$$E_\alpha = \{x : \dim_{\text{loc}}(\mu, x) = \alpha\}.$$

It is common to say that “ μ satisfies the multifractal formalism” if for all $\alpha \geq 0$ the Hausdorff and packing dimensions of E_α are given by the Legendre transform of the L^q -spectrum τ_q , that is,

$$\dim_H(E_\alpha) = \dim_p(E_\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu)\}. \quad (1.1)$$

See Section 2 below for the precise definitions.

Since its origins in physics literature in the 80’s (e.g. [8, 7]), the multifractal analysis has gained a lot of interest. For many relevant works related to multifractal formalism, see e.g. references in [6]. However, it seems that most of the studies take place in Euclidean spaces, or in spaces having a Euclidean type manifold structure.

In this paper, our goal is to study the local dimensions of measures in doubling metric spaces. Perhaps the most classical situation in which the multifractal formalism is known to hold is the case of self-similar measures in Euclidean spaces under the strong separation condition; see e.g. [3, 5]. Our main results can be viewed as a generalisation of this result into metric spaces, but our method using local versions of the L^q -spectrum and dimensions is useful also in the Euclidean setting.

In a general doubling metric space, there are usually no nontrivial self-similar maps, but often there is still a large class of Moran constructions sharing many of the geometric properties of self-similar iterated function systems. We will consider measures on the limit sets of these Moran constructions and investigate the behaviour and multifractality of $\dim_{\text{loc}}(\mu, x)$ for these measures. To determine $\dim_{\text{loc}}(\mu, x)$, we consider the local L^q -spectrum of μ . As for the classical (global) spectrum, the definition involves sums of the form $\sum_{B \in \mathcal{B}} \mu(B)^q$ over packings or partitions of the space X . However, in order to make the notion local, only those $B \in \mathcal{B}$ are taken into account which are “sufficiently close to x ”. It turns out that in many cases, the local spectrum gives more precise information on $\dim_{\text{loc}}(\mu, x)$ than its global counterpart. The local spectrum was introduced

Date: September 18, 2012.

2000 Mathematics Subject Classification. Primary 28A80; Secondary 28D20, 54E50.

Key words and phrases. local L^q -spectrum, local multifractal formalism.

The authors acknowledge the support of the Academy of Finland, projects #114821, #126976, #137528 and #211229.

in [9] as a tool to study local homogeneity properties of measures. Although the definition seems very natural, we were not able to track a definition of a local spectrum for measures in the existing literature. In [1], a local spectrum for functions is defined in order to study their Hölder regularity.

The paper is organised as follows. In Section 2, we set up some notation and define the necessary concepts. Further, in Section 3 we consider partitions of the space X , and show how the various dimensions and dimension spectra can be calculated using these partitions. We also relate the L^q -spectra and dimensions to the local entropy dimensions also defined in [9]. Our main results are presented in Section 4. We first give a series of conditions for Moran constructions in doubling metric spaces and measures defined on their limit sets. Then we show how the local L^q -spectrum can be used to calculate the local dimensions of these measures, and finally study their multifractal properties.

2. NOTATION AND PRELIMINARIES

In this paper, we always assume the metric space (X, d) to be *doubling*, meaning that there is a constant $N = N(X) \in \mathbb{N}$, called the *doubling constant* of X , such that any closed ball $B(x, r) = \{y \in X : d(x, y) \leq r\}$ with centre $x \in X$ and radius $r > 0$ can be covered by N balls of radius $r/2$.

For $M > 0$ and a ball $B = B(x, r)$, we will use the abbreviation $MB = B(x, Mr)$. For this to make sense, we always assume that the radius and centre of the ball B have been fixed, even if these are not explicitly mentioned.

We call any countable collection \mathcal{B} of pairwise disjoint closed balls a *packing*. It is called a *packing of A* for a subset $A \subset X$ if the centers of the balls of \mathcal{B} are in the set A , and it is a δ -packing (for $\delta > 0$) if all the balls in \mathcal{B} have radius δ . A δ -packing \mathcal{B} of A is termed *maximal* if for every $x \in A$ there is $B \in \mathcal{B}$ so that $B(x, \delta) \cap B \neq \emptyset$. Note that if \mathcal{B} is a maximal δ -packing of A , then $2\mathcal{B} = \{2B : B \in \mathcal{B}\}$ covers A .

The following lemma will be frequently used in this paper.

Lemma 2.1. *For a metric space X , the following statements are equivalent:*

- (1) *X is doubling.*
- (2) *There are $s > 0$ and $c > 0$ such that for all $R > r > 0$ any ball of radius R can be covered by $c(r/R)^{-s}$ balls of radius r .*
- (3) *There are $s > 0$ and $c > 0$ such that if $R > r > 0$ and \mathcal{B} is an r -packing of a closed ball of radius R , then the cardinality of \mathcal{B} is at most $c(r/R)^{-s}$.*
- (4) *For every $0 < \lambda < 1$ there is a constant $M = M(X, \lambda) \in \mathbb{N}$, satisfying the following: If \mathcal{B} is a collection of closed balls of radius $\delta > 0$ so that $\lambda\mathcal{B}$ is pairwise disjoint, then there are δ -packings $\{\mathcal{B}_1, \dots, \mathcal{B}_M\}$ so that $\mathcal{B} = \bigcup_{i=1}^M \mathcal{B}_i$.*
- (5) *There is $M = M(X) \in \mathbb{N}$ such that if $A \subset X$ and $\delta > 0$, then there are δ -packings of A , $\mathcal{B}_1, \dots, \mathcal{B}_M$ whose union covers A .*

The upper and lower local dimensions of a measure μ at x are defined by

$$\overline{\dim}_{\text{loc}}(\mu, x) = \limsup_{r \downarrow 0} \log \mu(B(x, r)) / \log r,$$

$$\underline{\dim}_{\text{loc}}(\mu, x) = \liminf_{r \downarrow 0} \log \mu(B(x, r)) / \log r,$$

respectively. If the upper and lower dimensions agree, we call their mutual value the *local dimension of the measure μ at x* and write $\dim_{\text{loc}}(\mu, x)$ for this common value. In this article, a *measure* exclusively refers to a nontrivial Borel regular (outer) measure defined on all subsets of X so that bounded sets have finite measure.

For estimating the local dimensions we will use the local L^q -dimensions defined in [9]. Although the local definitions are obtained via their global counterparts in small balls, their behaviour can be quite different; see [9, Examples 5.1–5.2].

Let μ be a measure on X , $A \subset X$ a bounded set and $q \in \mathbb{R}$. The (*global*) L^q -spectrum of μ on A is defined by

$$\tau_q(\mu, A) = \liminf_{\delta \downarrow 0} \frac{\log S_q(\mu, A, \delta)}{\log \delta},$$

where

$$S_q(\mu, A, \delta) = \sup \left\{ \sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } A \cap \text{spt}(\mu) \right\} \quad (2.1)$$

is the L^q -moment sum of μ on A at the scale δ . Note that if $q \geq 0$, the definition of $\tau_q(\mu, A)$ does not change if $A \cap \text{spt}(\mu)$ is replaced by A in the right-hand side of (2.1). If $q \neq 1$, then we define the (*global*) L^q -dimension of μ on A by setting

$$\dim_q(\mu, A) = \tau_q(\mu, A)/(q - 1).$$

We also denote $\tau_q(\mu) = \tau_q(\mu, X)$ and $\dim_q(\mu) = \dim_q(\mu, X)$ provided that X is bounded.

In the case $q = 1$ the above definition make no sense. Thus we define for every $A \subset X$ with $\mu(A) > 0$ the (*global*) upper and lower entropy dimensions of μ on A as

$$\begin{aligned} \overline{\dim}_1(\mu, A) &= \limsup_{\delta \downarrow 0} \int_A \frac{\log \mu(B(y, \delta))}{\log \delta} d\mu(y), \\ \underline{\dim}_1(\mu, A) &= \liminf_{\delta \downarrow 0} \int_A \frac{\log \mu(B(y, \delta))}{\log \delta} d\mu(y), \end{aligned}$$

respectively. If they agree, then their common value is denoted by $\dim_1(\mu, A)$. Here and hereafter, for $A \subset X$ and a μ -measurable $f: X \rightarrow \overline{\mathbb{R}}$, we use the notation $\int_A f(y) d\mu(y) = \mu(A)^{-1} \int_A f(y) d\mu(y)$ whenever the integral is well defined.

From the above global definitions we then derive their local versions. The *local* L^q -spectrum of μ at $x \in \text{spt}(\mu)$ is defined as

$$\tau_q(\mu, x) = \lim_{r \downarrow 0} \tau_q(\mu, B(x, r))$$

and the *local* L^q -dimension of μ at x as

$$\dim_q(\mu, x) = \lim_{r \downarrow 0} \dim_q(\mu, B(x, r)) = \tau_q(\mu, x)/(q - 1).$$

Correspondingly, the *local* upper and lower entropy dimensions at $x \in \text{spt}(\mu)$ are defined as

$$\begin{aligned} \overline{\dim}_1(\mu, x) &= \limsup_{r \downarrow 0} \overline{\dim}_1(\mu, B(x, r)), \\ \underline{\dim}_1(\mu, x) &= \liminf_{r \downarrow 0} \underline{\dim}_1(\mu, B(x, r)). \end{aligned}$$

For the basic properties of \dim_q , we refer to [9].

The following theorem lists the main relationships between the different local dimensions. Recall that a measure μ has the density point property, if

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1$$

for μ -almost all $x \in A$ whenever $A \subset X$ is μ -measurable.

Theorem 2.2. *If μ is a measure on a doubling metric space X , then*

$$\lim_{q \downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_{\text{loc}}(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \leq \lim_{q \uparrow 1} \dim_q(\mu, x) \quad (2.2)$$

for μ -almost all $x \in X$ and

$$\lim_{q \downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \lim_{q \uparrow 1} \dim_q(\mu, x) \quad (2.3)$$

for every $x \in \text{spt}(\mu)$.

Furthermore, if the measure μ has the density point property, then

$$\underline{\dim}_{\text{loc}}(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \quad (2.4)$$

for μ -almost all $x \in X$.

The claims (2.2) and (2.4) are proved in [9, Theorem 3.1 and Theorem 3.11] and (2.3) follows immediately from Proposition 3.7 below. It is worthwhile to notice that the density point property is not needed in the global version of (2.4) whereas in the local case, it is a necessary assumption; see [9, Remark 3.12 and Examples 5.7–5.8].

3. ENTROPY AND L^q -DIMENSIONS USING PARTITIONS

In this section, we reformulate the main definitions using partitions of the space X and show that these definitions are consistent with the ordinary definitions presented above. The use of the partitions is motivated by the fact that the original definition of the L^q -dimension using packings often causes technical problems if $q < 0$. Moreover, the measures that we are interested in usually have some additional a priori structure for which the partition definition suits well. For instance, see Lemma 4.1. We also use the partition definitions to relate the L^q - and entropy dimensions in Proposition 3.7.

Let $1 \leq \Lambda < \infty$. A countable partition \mathcal{Q} of X is called a (δ, Λ) -partition (for $\delta > 0$) if all the sets of \mathcal{Q} are Borel sets and for each $Q \in \mathcal{Q}$ there exists a ball B_Q so that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}\}$ is a δ -packing. The choice of Λ is usually not important, and thus we simply talk about δ -partitions and assume that Λ has been silently fixed. Usually we consider δ_n -partitions for a sequence of δ_n and in this case we assume that Λ is the same for all δ_n .

Let $(\delta_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive real numbers so that there is $0 < c < 1$ for which

$$\delta_n < c^n \quad (3.1)$$

for all n and

$$\log \delta_n / \log \delta_{n+1} \longrightarrow 1 \quad (3.2)$$

as $n \rightarrow \infty$. For each $n \in \mathbb{N}$ we fix a δ_n -partition \mathcal{Q}_n . If $x \in X$, then we denote the unique element of \mathcal{Q}_n containing x by $Q_n(x)$. Furthermore, if $A \subset X$, then we set $\mathcal{Q}_n(A) = \{Q \in \mathcal{Q}_n : A \cap Q \neq \emptyset\}$ for all $n \in \mathbb{N}$.

Perhaps the most classical example of a δ -partition is the dyadic cubes of the Euclidean space. We remark that in doubling metric spaces, it is possible to define similar kind of nested partitions sharing most of the good properties of dyadic cubes; see [10] and references therein. But often in applications, the nested structure is inconvenient to work with. Since the δ_n -partitions do not have to be nested, they are slightly more flexible than such generalised nested cubes.

Throughout this section, we assume that for each $n \in \mathbb{N}$ we have a fixed δ_n -partition \mathcal{Q}_n , where $(\delta_n)_{n \in \mathbb{N}}$ is a decreasing sequence satisfying (3.1) and (3.2).

3.1. Local dimensions via partitions. We include a proof of the following folklore result, Proposition 3.1, since we have not been able to track a complete proof in the literature (see e.g. [4, Lemma 2.3] and [11, Theorem 15.3]).

To simplify the notation, we set

$$\begin{aligned} \overline{D}_{\text{loc}}(\mu, x) &= \limsup_{n \rightarrow \infty} \log \mu(Q_n(x)) / \log \delta_n, \\ \underline{D}_{\text{loc}}(\mu, x) &= \liminf_{n \rightarrow \infty} \log \mu(Q_n(x)) / \log \delta_n \end{aligned}$$

for all measures μ on X and $x \in X$. A priori, the definitions of $\overline{D}_{\text{loc}}(\mu, x)$ and $\underline{D}_{\text{loc}}(\mu, x)$ depend on the choice of the partition, but Proposition 3.1 implies that almost everywhere these quantities equal the local dimensions and hence, the choice of the partition does not play any role.

Proposition 3.1. *If μ is a measure on a doubling metric space X , then*

$$\begin{aligned} \overline{\dim}_{\text{loc}}(\mu, x) &= \overline{D}_{\text{loc}}(\mu, x), \\ \underline{\dim}_{\text{loc}}(\mu, x) &= \underline{D}_{\text{loc}}(\mu, x) \end{aligned}$$

for μ -almost all $x \in X$.

Proof. The inequalities $\overline{\dim}_{\text{loc}}(\mu, x) \leq \overline{D}_{\text{loc}}(\mu, x)$, $\underline{\dim}_{\text{loc}}(\mu, x) \leq \underline{D}_{\text{loc}}(\mu, x)$ are seen to hold for all $x \in X$ by using (3.2) and the fact $Q_n(x) \subset B(x, (\Lambda + 1)\delta_n)$ for all $x \in X$ and $n \in \mathbb{N}$.

To prove the estimates in the other direction, fix a bounded set $A \subset X$, $0 < t < s < \infty$ and define

$$A_n(t, s) = \{x \in A : \mu(Q_n(x)) < \delta_n^s \text{ and } \mu(B(x, \delta_n)) > \delta_n^t\}.$$

Now the set

$$\{x \in A : \overline{\dim}_{\text{loc}}(\mu, x) < \overline{D}_{\text{loc}}(\mu, x) \text{ or } \underline{\dim}_{\text{loc}}(\mu, x) < \underline{D}_{\text{loc}}(\mu, x)\}$$

is contained in

$$\bigcup_{0 < t < s < \infty} \bigcap_{k \in \mathbb{N}} \bigcup_{n=k}^{\infty} A_n(t, s)$$

where the union is over countably many (e.g. rational) t and s . Thus, by the Borel-Cantelli Lemma, it suffices to show that $\sum_{n \in \mathbb{N}} \mu(A_n(t, s)) < \infty$ for any choice of t and s . To verify this, let $n \in \mathbb{N}$ and consider $x \in A_n(t, s)$. Since only at most $C = C(N, \Lambda)$ of the sets $Q \in \mathcal{Q}_n$ meet $B(x, \delta_n)$ (cf. Lemma 2.1(3)), we have the estimate $\mu(A_n(t, s) \cap B(x, \delta_n)) \leq C\delta_n^s = C\delta_n^{s-t}\delta_n^t \leq C\delta_n^{s-t}\mu(B(x, \delta_n))$. Using Lemma 2.1(5), we may cover $A_n(t, s)$ by a union of at most $M = M(N)$ δ_n -packings of $A_n(t, s)$ and thus

$$\mu(A_n(t, s)) \leq CM\delta_n^{s-t}\mu(B),$$

where B is a ball centered at A with radius $\text{diam}(A) + 1$. By (3.1), the sum $\sum_{n \in \mathbb{N}} \delta_n^{s-t}$ converges and the claim holds for μ -almost all $x \in A$. As this is true for any bounded $A \subset X$, this finishes the proof. \square

3.2. L^q -spectrum and entropy dimension via partitions. The following proposition shows that both the local and global L^q -spectrum and L^q -dimension can equivalently be defined by using partitions. Later we will show that this is also the case for the local entropy dimension, see Proposition 3.4. For the global entropy dimension the situation is slightly more complicated.

Proposition 3.2. *If μ is a measure on a doubling metric space X , $A \subset X$ is bounded with $\mu(A) > 0$ and $q \geq 0$, then*

$$\tau_q(\mu, A) = \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n}.$$

Proof. Let $0 < \delta < \delta_1$, and $n \in \mathbb{N}$ so that $\delta_{n+1} \leq \delta < \delta_n$. Our first goal is to show that for a constant $c_1 = c_1(N, \Lambda, q) > 0$, we have

$$S_q(\mu, A, \delta) \leq c_1 \left(\frac{\delta_n}{\delta} \right)^s \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q, \quad (3.3)$$

where $s = s(N) > 0$ is the constant given by Lemma 2.1(3). Recall that N is the doubling constant of X and Λ is the fixed constant used in defining the partitions \mathcal{Q}_n . To show (3.3), we fix a δ -packing \mathcal{B} of A and let

$$\mathcal{C}_B = \{Q \in \mathcal{Q}_n(A) : Q \cap B \neq \emptyset\}$$

for all $B \in \mathcal{B}$. Since \mathcal{C}_B is a cover for B , we have

$$\mu(B)^q \leq \left(\sum_{Q \in \mathcal{C}_B} \mu(Q) \right)^q \leq (\#\mathcal{C}_B)^q \sum_{Q \in \mathcal{C}_B} \mu(Q)^q,$$

where $\#\mathcal{C}_B$ is the cardinality of \mathcal{C}_B . Notice that all the sets of \mathcal{C}_B are contained in a ball of radius $(1 + 2\Lambda)\delta_n$ which, on the other hand, has a δ_n -packing of cardinality $\#\mathcal{C}_B$. Hence, Lemma 2.1(3) implies that $\#\mathcal{C}_B \leq c_2 = c_2(N, \Lambda)$ for all $B \in \mathcal{B}$ and therefore

$$\sum_{B \in \mathcal{B}} \mu(B)^q \leq c_2^q \sum_{B \in \mathcal{B}} \sum_{Q \in \mathcal{C}_B} \mu(Q)^q.$$

Furthermore, by Lemma 2.1(3) there exists a constant $c_3 = c_3(N, \Lambda) > 0$ so that the cardinality of the set $\{B \in \mathcal{B} : Q \cap B \neq \emptyset\}$ is at most $c_3(\delta_n/\delta)^s$ for all $Q \in \mathcal{Q}_n$. Thus, (3.3) follows with $c_1 = c_2^q c_3$.

To find an estimate in the other direction, choose for each $Q \in \mathcal{Q}_n(A)$ a point $x_Q \in A \cap Q$ and a ball B_Q so that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a δ_n -packing. Notice that $Q \subset B(x_Q, 2\Lambda\delta_n) \subset 3\Lambda B_Q$ for all $Q \in \mathcal{Q}_n(A)$. According to Lemma 2.1(4) there exists $M = M(N, \Lambda) \in \mathbb{N}$ and $\mathcal{Q}_1, \dots, \mathcal{Q}_M$ so that $\mathcal{Q}_n(A) = \bigcup_{i=1}^M \mathcal{Q}_i$ and $\{3\Lambda B_Q : Q \in \mathcal{Q}_i\}$ is a $3\Lambda\delta_n$ -packing for all $i \in \{1, \dots, M\}$. Thus $\{B(x_Q, 2\Lambda\delta_n) : Q \in \mathcal{Q}_i\}$ is a $2\Lambda\delta_n$ -packing of A for all $i \in \{1, \dots, M\}$. Since $\bigcup_{Q \in \mathcal{Q}_n(A)} Q \subset \bigcup_{i=1}^M \bigcup_{Q \in \mathcal{Q}_i} B(x_Q, 2\Lambda\delta_n)$, we may choose $i \in \{1, \dots, M\}$ so that

$$\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q \leq M^{-1} \sum_{Q \in \mathcal{Q}_i} \mu(B(x_Q, 2\Lambda\delta_n))^q \leq M^{-1} S_q(\mu, A, 2\Lambda\delta_n). \quad (3.4)$$

The proof now follows by combining (3.3) and (3.4) and taking logarithms and limits. \square

The global entropy dimension can be defined via partitions if A is compact. Before showing this, we exhibit a small technical lemma.

Lemma 3.3. *Suppose μ is a measure on a doubling metric space X and $A \subset X$ is bounded. Let $s > 0$ and $c > 0$ be as in Lemma 2.1(3). Then*

$$\int_A \log \mu(B(y, \delta)) d\mu(y) \geq -\frac{1}{e} - \mu(A) \left(\log c + s \log \frac{4 \operatorname{diam}(A)}{\delta} \right)$$

for all $\delta > 0$.

Proof. Let \mathcal{B}' be a maximal $\delta/4$ -packing of A and let $\mathcal{B} = \frac{1}{4}\mathcal{B}' = \{B_1, B_2, \dots, B_k\}$. Define $Q_1 = 8B_1 \setminus \bigcup_{B \in \mathcal{B} \setminus \{B_1\}} B$ and

$$Q_{n+1} = \left(8B_{n+1} \setminus \bigcup_{B \in \mathcal{B} \setminus \{B_{n+1}\}} B \right) \setminus \bigcup_{i=1}^n Q_i$$

for all $1 \leq n \leq k-1$. Then $\mathcal{Q} = \{Q_1 \cap A, Q_2 \cap A, \dots, Q_k \cap A\}$ is a $(\delta/16, 8)$ -partition of A in the relative metric.

If the unique $Q \in \mathcal{Q}$ containing $y \in A$ is denoted by $Q(y)$, then we have $Q(y) \subset B(y, \delta)$ for all $y \in A$. By Theorem 2.1(3) there exist constants $s > 0$ and $c > 0$ depending only on the doubling constant of A so that $\#\mathcal{Q} \leq c(4 \operatorname{diam}(A)/\delta)^s$. Now Jensen's inequality gives

$$\begin{aligned} \int_A \log \mu(B(y, \delta)) d\mu(y) &\geq \int_A \log \mu(Q(y)) d\mu(y) = \sum_{Q \in \mathcal{Q}} \mu(Q) \log \mu(Q) \\ &\geq \mu(A) \log \frac{\mu(A)}{\#\mathcal{Q}} \geq \mu(A) \log \mu(A) - \mu(A) \log \frac{c(4 \operatorname{diam}(A))^s}{\delta^s} \end{aligned}$$

and the claim follows. \square

Proposition 3.4. *If μ is a measure on a doubling metric space X and $A \subset X$ is compact with $\mu(A) > 0$, then*

$$\begin{aligned} \overline{\dim}_1(\mu, A) &= \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n}, \\ \underline{\dim}_1(\mu, A) &= \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n}. \end{aligned}$$

Proof. Choose for each $Q \in \mathcal{Q}_n(A)$ a ball B_Q such that $Q \subset \Lambda B_Q$ and $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a δ_n -packing. If $Q \in \mathcal{Q}_n(A)$, then for every $y \in Q$ we have

$$Q \subset B(y, 2\Lambda\delta_n) \subset 3\Lambda B_Q \subset \bigcup_{Q' \in \mathcal{C}_Q} Q',$$

where $\mathcal{C}_Q = \{Q' \in \mathcal{Q}_n(A) : Q' \cap 3\Lambda B_Q \neq \emptyset\}$. Thus, letting $A_n = \bigcup_{Q \in \mathcal{Q}_n(A)} Q$, we get

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) &\leq \sum_{Q \in \mathcal{Q}_n(A)} \int_Q \log \mu(B(y, 2\Lambda\delta_n)) d\mu(y) \\ &= \int_{A_n} \log \mu(B(y, 2\Lambda\delta_n)) d\mu(y) \leq \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \sum_{Q' \in \mathcal{C}_Q} \mu(Q'). \end{aligned}$$

Moreover, since each $Q' \in \mathcal{Q}_n(A)$ is contained in at most $c_4(3\Lambda)^s$ collections \mathcal{C}_Q by Lemma 2.1(3), where $c_4 = c_4(N) < \infty$, we have

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \sum_{Q' \in \mathcal{C}_Q} \mu(Q') - \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) \\ &= \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \left(1 + \frac{\sum_{Q' \in \mathcal{C}_Q \setminus \{Q\}} \mu(Q')}{\mu(Q)} \right) \\ &\leq \sum_{Q \in \mathcal{Q}_n(A)} \sum_{Q' \in \mathcal{C}_Q \setminus \{Q\}} \mu(Q') \leq c_4(3\Lambda)^s \mu(B_0), \end{aligned}$$

where B_0 is a ball centered at A with radius $\text{diam}(A) + 2\Lambda\delta_n$. Putting these estimates together, we get

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) &\leq \int_{A_n} \log \mu(B(y, \delta)) d\mu(y) \\ &\leq \sum_{Q \in \mathcal{Q}_{n-1}(A)} \mu(Q) \log \mu(Q) + c_4(3\Lambda)^s \mu(B_0) \end{aligned} \tag{3.5}$$

for all $2\Lambda\delta_n \leq \delta \leq 2\Lambda\delta_{n-1}$.

Since A is compact we have $\lim_{n \rightarrow \infty} \mu(A_n \setminus A) = 0$ and therefore, by Lemma 3.3,

$$\lim_{n \rightarrow \infty} \frac{1}{\log \delta_n} \int_{A_n \setminus A} \log \mu(B(y, 2\Lambda\delta_n)) d\mu(y) = 0.$$

From this, (3.2) and (3.5) the claim follows easily. \square

Example 3.5. In this example, we show that the claim in Proposition 3.4 does not hold for non-compact sets. Equip $X = [0, 1]$ with the Euclidean metric and let \mathcal{Q}_n be the partition of X to the dyadic intervals of length 2^{-n} . Let $A = \mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$ and $\nu = \sum_{i=1}^{\infty} 2^{-i} \delta_{q_i}$, where δ_x denotes the Dirac unit mass located at x . Finally, set $\mu = \mathcal{L}^1|_{[0,1]} + \nu$, where \mathcal{L} denotes Lebesgue measure.

Since $\sum_{Q \in \mathcal{Q}_n} \mu(Q) \log \mu(Q) \leq \log 2^{-n}$ for all n large enough, we have

$$\liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log 2^{-n}} = \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n} \mu(Q) \log \mu(Q)}{2 \log 2^{-n}} \geq \frac{1}{2}.$$

Let $\varepsilon > 0$, choose $k \in \mathbb{N}$ so that $\sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon$, and define $A' = \{q_1, \dots, q_k\}$. According to Lemma 3.3, there exists $c > 0$ so that

$$\begin{aligned} \int_{A'} \log \mu(B(y, \delta)) d\mu(y) &\geq -\frac{1}{e} - \nu(A') k \log 2, \\ \int_{A \setminus A'} \log \mu(B(y, \delta)) d\mu(y) &\geq -\frac{1}{e} - \nu(A \setminus A') (\log c + \log(4/\delta)) \end{aligned}$$

for all $\delta > 0$ small enough. Since $\nu(A \setminus A') < \varepsilon$, we get

$$\overline{\dim}_1(\mu, A) = \limsup_{\delta \downarrow 0} \int_A \frac{\log \mu(B(y, \delta))}{\log \delta} \leq \limsup_{\delta \downarrow 0} \frac{-\frac{2}{e} - \nu(A') k \log 2 - \varepsilon \log(4c) + \varepsilon \log \delta}{\log \delta} = \varepsilon.$$

Thus $\overline{\dim}_1(\mu, A) = 0$.

Remark 3.6. In view of the definitions of \dim_q , it is natural to ask if the entropy dimensions could also be defined in terms of maximal packings. However, simple examples such as $\mu = \mathcal{L}^1|_{[0,1]} + \delta_1$ on $[0, 1]$ show that this is usually not possible.

To finish this section, we show that the definition of the entropy dimension as \dim_1 is consistent with the monotonicity of the L^q -dimensions.

Proposition 3.7. *If μ is a measure on a doubling metric space X and $A \subset X$ compact with $\mu(A) > 0$, then*

$$\lim_{q \downarrow 1} \dim_q(\mu, A) \leq \underline{\dim}_1(\mu, A) \leq \overline{\dim}_1(\mu, A) \leq \lim_{q \uparrow 1} \dim_q(\mu, A).$$

Proof. The existence of the limits follows from [9, Proposition 2.7]. Thus, the claims follow if we can show that

$$\tau_q(\mu, A)/(q - 1) \geq \overline{\dim}_1(\mu, A) \geq \underline{\dim}_1(\mu, A) \geq \tau_p(\mu, A)/(p - 1), \quad (3.6)$$

where $0 < q < 1 < p$. Define $h_n(q) = \log \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q$ for all $q \geq 0$. A simple application of Hölder's inequality shows that h_n is convex. As $\mathcal{Q}_n(A)$ has only a finite number of elements, h_n is differentiable with $h'_n(1) = (\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q))^{-1} \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)$. Thus

$$\frac{h_n(q) - h_n(1)}{q - 1} \leq h'_n(1) \leq \frac{h_n(p) - h_n(1)}{p - 1}.$$

Using these estimates and the fact that $h_n(1) = \log \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)$ does not depend on n , we calculate

$$\begin{aligned} \frac{1}{q - 1} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n} &= \limsup_{n \rightarrow \infty} \frac{h_n(q) - h_n(1)}{(q - 1) \log \delta_n} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n} \geq \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n} \\ &\geq \liminf_{n \rightarrow \infty} \frac{h_n(p) - h_n(1)}{(p - 1) \log \delta_n} = \frac{1}{p - 1} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^p}{\log \delta_n}. \end{aligned}$$

The desired estimate (3.6) now follows from Propositions 3.2 and 3.4. \square

4. LOCAL DIMENSION AND MULTIFRACTAL ANALYSIS FOR MORAN MEASURES

We now turn towards our final goal to study multifractality of measures in metric spaces. We first introduce a class of Moran constructions in a complete doubling metric space X and show how Theorem 2.2 can be applied to calculate the local dimensions for a large class of measures defined on these Moran fractals. Then, under certain additional assumptions, we turn to study the multifractal spectrum of these measures. Our main aim is to show that using the technique introduced in [9], we can push the standard methods used to calculate the local dimensions for self-similar measures on Euclidean spaces (see [3, 13, 5]) to obtain analogous results in doubling metric spaces with very mild regularity assumptions, see Remark 4.3.

4.1. Moran constructions and measures. Let $m \in \mathbb{N}$, $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$, $\Sigma_n = \{1, \dots, m\}^n$ for all $n \in \mathbb{N}$, and $\Sigma_* = \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} \Sigma_n$. If $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma \cup \bigcup_{j=n}^{\infty} \Sigma_j$, then we let $\mathbf{i}|_n = (i_1, \dots, i_n)$ (and $\mathbf{i}|_0 = \emptyset$). The concatenation of two words $\mathbf{i} \in \Sigma_*$ and $\mathbf{j} \in \Sigma \cup \Sigma_*$ is denoted by $\mathbf{i}\mathbf{j}$. We also set $\mathbf{i}^- = \mathbf{i}|_{n-1}$ for $\mathbf{i} \in \Sigma_n$ and $n \in \mathbb{N}$. By $|\mathbf{i}|$, we denote the length of a word $\mathbf{i} \in \Sigma_*$. We assume that $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ is a collection of compact subsets of X that satisfy the following conditions for some constants $0 < C_0, C_1 < \infty$:

- (M1) $E_{\mathbf{i}} \subset E_{\mathbf{i}^-}$ for all $\emptyset \neq \mathbf{i} \in \Sigma_*$.
- (M2) $E_{\mathbf{i}\mathbf{i}} \cap E_{\mathbf{i}\mathbf{j}} = \emptyset$ if $\mathbf{i} \in \Sigma_*$ and $i \neq j$.

- (M3) For each $\mathbf{i} \in \Sigma_*$, there is $x \in E_{\mathbf{i}}$ such that $B(x, C_0 \operatorname{diam}(E_{\mathbf{i}})) \subset E_{\mathbf{i}}$.
- (M4) $\operatorname{diam}(E_{\mathbf{i}|n}) \rightarrow 0$ as $n \rightarrow \infty$, for each $\mathbf{i} \in \Sigma$.
- (M5) $\operatorname{diam}(E_{\mathbf{i}-})/\operatorname{diam}(E_{\mathbf{i}}) \leq C_1 < \infty$ for all $\emptyset \neq \mathbf{i} \in \Sigma_*$.

We define the limit set of the construction as $E = \bigcap_{n \in \mathbb{N}} \bigcup_{\mathbf{i} \in \Sigma_n} E_{\mathbf{i}}$ and given $\mathbf{i} \in \Sigma$, denote by $x_{\mathbf{i}}$ the point obtained as $\{x_{\mathbf{i}}\} = \bigcap_{n \in \mathbb{N}} E_{\mathbf{i}|n}$. We assume that for each $i \in \{1, \dots, m\}$ there is a continuous function $r_i: E \rightarrow (0, 1)$. Given $x \in E$, and $\mathbf{i} \in \Sigma_n$, we let $r_{\mathbf{i}}(x) = \prod_{k=1}^n r_{i_k}(x)$. Moreover, we assume that

- (M6) $\lim_{n \rightarrow \infty} \log \operatorname{diam}(E_{\mathbf{i}|n}) / \log r_{\mathbf{i}|n}(x_{\mathbf{i}}) = 1$ uniformly for all $\mathbf{i} \in \Sigma$.

The following further conditions on the collection $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ are needed only in subsection 4.3. For $n \in \mathbb{N}$, denote $\mathcal{E}_n = \{E_{\mathbf{i}} : \operatorname{diam}(E_{\mathbf{i}}) \leq C_1/(C_0 2^n) < \operatorname{diam}(E_{\mathbf{i}-})\}$.

- (M7) There is $c > 0$ so that for each $Q \in \mathcal{E}_n$, there is $x \in E$ with $B(x, c2^{-n}) \subset Q$.

- (M8) $\lim_{r \downarrow 0} \log r / \log(\operatorname{diam}(E_{\mathbf{i}|n(\mathbf{i}, r)})) = 1$ for all $\mathbf{i} \in \Sigma$, where $n(\mathbf{i}, r) = \max\{n \in \mathbb{N} : B(x_{\mathbf{i}}, r) \cap E \subset E_{\mathbf{i}|n}\}$.

Our next lemma shows how we can obtain a δ -partition of X from the elements of $\{E_{\mathbf{i}}\}$ which are roughly of size δ .

Lemma 4.1. *If $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ is a collection of compact sets satisfying the conditions (M1)–(M5), then for every $n \in \mathbb{N}$ there is a (2^{-n}) -partition \mathcal{Q}_n of X such that each $E_{\mathbf{i}} \in \mathcal{E}_n$ is a subset of some $Q \in \mathcal{Q}_n$ and all elements of \mathcal{Q}_n contain at most one element of \mathcal{E}_n .*

Proof. Consider a maximal collection \mathcal{B}_n of disjoint balls of radius 2^{-n} contained in $X \setminus \bigcup \mathcal{E}_n$. Define $\mathcal{A}_n = \mathcal{E}_n \cup \mathcal{B}_n = \{A_1, A_2, \dots\}$. For each $x \in X$, we let $i_x = \min\{j \in \mathbb{N} : \operatorname{dist}(x, A_j) = \min_{A \in \mathcal{A}_n} \operatorname{dist}(x, A)\}$ and set $Q_{A_i} = \{x \in X : i_x = i\}$ for all $i \in \mathbb{N}$. It is then easy to see that $\mathcal{Q}_n = \{Q_A : A \in \mathcal{A}_n\}$ is the desired (2^{-n}) -packing. Observe that each $Q \in \mathcal{Q}_n$ is a Borel set since $\bigcup_{i=1}^k Q_{A_i}$ is closed for all k . Moreover, the constant Λ of this partition depends only on the constants C_0 and C_1 as one may choose $\Lambda = C_0 C_1 + 1$. \square

Let μ be a probability measure on X with $\operatorname{spt}(\mu) = E$. Then for each $\mathbf{i} \in \Sigma_*$, μ induces a probability vector $p_{\mathbf{i}} = (p_{\mathbf{i}}^1, \dots, p_{\mathbf{i}}^m)$ with $p_{\mathbf{i}}^i > 0$ for $i \in \{1, \dots, m\}$ such that $\mu(E_{\mathbf{i}i}) = p_{\mathbf{i}}^i \mu(E_{\mathbf{i}})$ for $i \in \{1, \dots, m\}$. Given $\mathbf{i} \in \Sigma_n$, we denote $\mu_{\mathbf{i}} := \mu(E_{\mathbf{i}}) = \prod_{j=1}^n p_{\mathbf{i}|j-1}^{i_j}$. In the next theorem, we assume that the weights $p_{\mathbf{i}}$ are controlled in terms of continuous probability functions $p(x) = (p_1(x), \dots, p_m(x))$. More precisely, we assume that for each $i \in \{1, \dots, m\}$, the function $p_i: E \rightarrow (0, 1)$ is continuous with $\sum_{i=1}^m p_i(x) = 1$ for all $x \in E$ and that for some $a > 0$, we have $p_i(x) > a$ for all x and i . Similarly to r_i , we define $P_{\mathbf{i}}(x) = \prod_{k=1}^n p_{i_k}(x)$ when $\mathbf{i} \in \Sigma_n$.

4.2. Local L^q -spectrum for Moran measures.

Theorem 4.2. *Let $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M6). Suppose that μ is a probability measure on E and let $p_{\mathbf{i}}$ and p be as above. If $p_{\mathbf{i}|n} \rightarrow p(x_{\mathbf{i}})$ as $n \rightarrow \infty$ uniformly for all $\mathbf{i} \in \Sigma$, then, for all $x \in E$ and all $q \geq 0$, $\tau_q(\mu, x)$ is the unique $\tau \in \mathbb{R}$ that satisfies*

$$\sum_{i=1}^m p_i(x)^q r_i(x)^{-\tau} = 1. \quad (4.1)$$

Moreover,

$$\dim_1(\mu, x) = \dim_{\text{loc}}(\mu, x) = \frac{\sum_{i=1}^m p_i(x) \log p_i(x)}{\sum_{i=1}^m p_i(x) \log r_i(x)} \quad (4.2)$$

for μ -almost all $x \in E$.

Proof. We prove the claim (4.1). The identities (4.2) then follow from (4.1) by implicit differentiation together with Theorem 2.2.

For each $n \in \mathbb{N}$, let \mathcal{Q}_n be as in Lemma 4.1. Given $\emptyset \neq \mathbf{i} \in \Sigma_*$, we denote by $Q_{\mathbf{i}}$ the unique element of $\bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$ that contains $E_{\mathbf{i}}$ and does not contain $E_{\mathbf{i}-}$ (we assume without loss

of generality that $\mathcal{E}_1 = \{E_\emptyset\}$ so that this makes sense for all n . Let us fix $q \geq 0$, $x \in E$ and let $i \in \Sigma$ so that $x = x_i$. Let τ be as in (4.1). We first prove that $\tau_q(\mu, x) \geq \tau$. Let $0 < c < 1$. Since $p_{i|n} \rightarrow p(x_i)$ uniformly and $y \mapsto p(y)$ is continuous, we may choose n_0 so large that $p_j^i > cp_i(x)$ whenever $i \in \{1, \dots, m\}$, $j \in \Sigma_*$, and $E_j \subset E_{i|n_0}$. Making n_0 even larger if necessary, we may also assume that

$$cr_i(y) \leq r_i(x) \leq r_i(y)/c \quad (4.3)$$

for all $y \in E_{i|n_0}$ and all $i \in \{1, \dots, m\}$.

Now, for all $r > 0$, we choose $N_0 \geq n_0$ so that $Q_j \subset B(x, r)$ whenever $j \in \Sigma_*$, and $E_j \subset E_{i|N_0}$. Given $n \geq N_0$, let $Z_n = \{j \in \Sigma_* : Q_j \in \mathcal{Q}_n \text{ and } E_j \subset E_{i|N_0}\}$. Let $\varepsilon_n = \min_{j \in Z_n} (\text{diam}(E_j)/r_j(x))^{-\tau}$. Now, denoting $c_0 = C_1^{-|\tau|} \mu_{i|N_0}^q$, we get an estimate

$$2^{n\tau} \sum_{j \in Z_n} \mu(Q_j)^q \geq C_1^{-|\tau|} \sum_{j \in Z_n} \mu_j^q \text{diam}(E_j)^{-\tau} \geq c_0 \varepsilon_n \sum_{j \in Z_n} c^{q|j|} P_j(x)^q r_j(x)^{-\tau}. \quad (4.4)$$

For each $j \in Z_n$, pick $y \in E_j$. Using (M6), we may assume that $\log r_j(y) \geq 2 \log \text{diam}(E_j)$ by making N_0 larger if necessary. Letting $r_{\max} = \max\{r_i(y) : y \in E \text{ and } i \in \{1, \dots, m\}\}$, we have

$$\log r_{\max}^{|j|} \geq \log r_j(y) \geq 2 \log \text{diam}(E_j) \geq -2n \log 2,$$

and consequently,

$$|j| \leq \frac{2 \log \text{diam}(E_j)}{\log r_{\max}} \leq \frac{-2n \log 2}{\log r_{\max}} \leq nC_2, \quad (4.5)$$

for a constant $C_2 < \infty$ independent of n . On the other hand,

$$\sum_{j \in Z_n} P_j(x)^q r_j(x)^{-\tau} = P_{i|n_0}(x)^q r_{i|n_0}(x)^{-\tau} =: C_3 \quad (4.6)$$

by iterative use of (4.1). Putting (4.4)–(4.6) together, we get

$$\log \sum_{j \in Z_n} \mu(Q_j)^q \geq \log(2^{-n\tau} c_0 \varepsilon_n c^{qnC_2} C_3) = \log 2^{-n\tau} + \log \varepsilon_n + qnC_2 \log c + \log(c_0 C_3). \quad (4.7)$$

To estimate $\log \varepsilon_n$, we choose $j \in Z_n$ such that $\varepsilon_n = (\text{diam}(E_j)/r_j(x))^{-\tau}$. Then

$$\log \varepsilon_n = -\tau \log \text{diam}(E_j) (1 - \log r_j(x)/\log \text{diam}(E_j)). \quad (4.8)$$

Moreover, $\log r_j(y) + |j| \log c \leq \log r_j(x) \leq \log r_j(y) - |j| \log c$ for all $y \in E_j$ by (4.3). Using (4.5), this gives

$$\frac{\log r_j(y)}{\log \text{diam}(E_j)} + C_4 \log c \leq \frac{\log r_j(x)}{\log \text{diam}(E_j)} \leq \frac{\log r_j(y)}{\log \text{diam}(E_j)} - C_4 \log c \quad (4.9)$$

for some constant $0 < C_4 < \infty$.

Using (4.7), (4.8), (4.9), and (M6), we finally get

$$\liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(B(x,r))} \mu(Q)^q}{\log 2^{-n}} \leq \liminf_{n \rightarrow \infty} \frac{\log \sum_{j \in Z_n} \mu(Q_j)^q}{\log 2^{-n}} \leq \tau - (qC_2 + |\tau|C_4) \log c / \log 2.$$

As $c < 1$ and $r > 0$ can be chosen arbitrarily, we get, by recalling Proposition 3.2, that $\tau_q(\mu, x) \leq \tau$.

To prove that $\tau_q(\mu, x) \geq \tau$, we first fix $0 < c < 1$ and $r_0 > 0$ so that $cp_j^i < p_i(x)$ and $cr_i(x) < r_i(y) < \frac{1}{c}r_i(x)$ whenever $i \in \{1, \dots, m\}$ and $y \in E_j \subset B(x, r_0)$. Then, if $0 < r < r_0$, we may find $n_0 \in \mathbb{N}$ and finitely many elements $E_k \in \mathcal{Q}_{n_0}$, $E_k \subset B(x, r_0)$ whose union covers $B(x, r)$. For each such E_k , and $n \geq n_0$, we put $Z_{n,k} = \{j \in \Sigma_* : Q_j \in \mathcal{Q}_n \text{ and } E_j \subset E_k\}$. Putting $M_n = \max_{j \in Z_{n,k}} (\text{diam}(E_j)/r_j(x))^{-\tau}$, we may estimate as in (4.4) to obtain

$$2^{n\tau} \sum_{j \in Z_{n,k}} \mu(Q_j)^q \leq C_5 M_n \sum_{j \in Z_{n,k}} c^{-q|j|} P_j(x)^q r_j(x)^{-\tau}.$$

Calculating as above, this implies

$$\liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(B(x,r))} \mu(Q)^q}{\log 2^{-n}} \geq \tau + (qC_2 + |\tau|C_4) \log c / \log 2.$$

Letting $r \downarrow 0$ and then $c \uparrow 1$, and using Proposition 3.2 gives $\tau_q(\mu, x) \geq \tau$. \square

Remark 4.3. (1) One can find Moran constructions that satisfy (M1)–(M6) on doubling metric spaces satisfying only mild regularity assumptions on the space X . For instance, it suffices to assume that the space is uniformly perfect. Different types of Moran constructions in metric spaces have been recently studied in [12].

(2) The result is interesting already in \mathbb{R}^n . We remark that a self-similar measure on a self-similar set satisfying the strong separation conditions is a model case for Theorem 4.2 in the special case when p_i and r_i are constant, see [5]. However, as p_i and r_i are allowed to vary depending on the point, Theorem 4.2 can be applied in more general situations.

4.3. Some multifractal analysis. To handle (1.1), we have to deal with τ_q for negative values of q and for this we use the following lemma. Observe that we cannot use Proposition 3.2 when $q < 0$.

Lemma 4.4. *Suppose that in the setting of Theorem 4.2 also (M7) holds. Then, for all $x \in E$, $\tau_q(\mu, x)$ is determined by (4.1) also when $q < 0$.*

Proof. Let $q < 0$, $x \in E$ and let $\tau \in \mathbb{R}$ be the unique solution of (4.1). With trivial modifications to the proof of Theorem 4.2, we see that

$$\tau = \lim_{t \downarrow 0} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_{n,t}} \mu(Q)^q}{\log 2^{-n}} \quad (4.10)$$

where $\mathcal{Q}_{n,t} = \{Q \in \mathcal{Q}_n : Q \subset B(x, t)\}$ and $Q \cap E \neq \emptyset$. (Observe that $\text{spt}(\mu) = E$.)

In order to prove that $\tau = \tau_q(\mu, x)$, let $t > 0$, $2^{-n} \leq \delta < 2^{-n+1} < t$ and $y \in E \cap B(x, t)$. Then there is $n_0 \in \mathbb{N}$ depending only on the numbers C_0 and C_1 so that $B(y, \delta) \supset Q_i$ for some $Q_i \in \mathcal{Q}_{n+n_0, 2t}$. Thus, for any δ -packing $\{B_i\}$ of $B(x, t) \cap \text{spt}(\mu)$, we have

$$\sum_i \mu(B_i)^q \leq \sum_{Q \in \mathcal{Q}_{n+n_0, 2t}} \mu(Q)^q. \quad (4.11)$$

To get an estimate in the other direction, we fix n and t and use the assumption (M7) to find for each $Q \in \mathcal{Q}_{n,t}$ a point $y \in \text{spt}(\mu) \cap B(x, t)$ such that for $B_Q = B(y, c2^{-n})$, we have $B_Q \subset Q$. Thus, for the $(c2^{-n})$ -packing $\{B_Q : Q \in \mathcal{Q}_{n,t}\}$, we have

$$\sum_{Q \in \mathcal{Q}_{n,t}} \mu(B_Q)^q \geq \sum_{Q \in \mathcal{Q}_{n,t}} \mu(Q)^q. \quad (4.12)$$

Combining (4.10)–(4.12), and taking logarithms, it follows that $\tau_q(\mu, x) = \tau$. \square

Remark 4.5. Inspecting the proofs of Theorem 4.2 and Lemma 4.4, we observe that in the setting of these results, $\liminf_{n \rightarrow \infty}$ in the definition of $\tau_q(\mu, x)$ can actually be replaced by $\lim_{n \rightarrow \infty}$.

Theorem 4.6. *Let $\{E_i : i \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M8). Suppose that $r = (r_1, \dots, r_m)$ and $p = (p_1, \dots, p_m)$ are constant functions and μ is a measure with $\text{spt}(\mu) = E$ such that $p_{i|n} \rightarrow p$ uniformly for all $i \in \Sigma$. If $0 \leq \alpha_{\min} \leq \alpha_{\max}$ are the asymptotic derivatives of the concave function $q \mapsto \tau_q(\mu)$, then (1.1) holds for all $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$.*

Remark 4.7. For the inhomogeneous Bernoulli products on $[0, 1]$, the claims of Theorems 4.2 and 4.6 have been obtained independently by Batakis and Testud [2, Corollary 1.3].

Theorem 4.6 follows from Lemma 4.8 below, by taking $\varepsilon = 0$. Observe that the mapping $q \mapsto \tau_q(\mu)$ is indeed concave and continuous on \mathbb{R} by inspecting (4.1). Moreover, one easily derives that $\alpha_{\min} = \min\{\frac{\log p_i}{\log r_i} : i \in \{1, \dots, m\}\}$ and $\alpha_{\max} = \max\{\frac{\log p_i}{\log r_i} : i \in \{1, \dots, m\}\}$. Furthermore, $\underline{\dim}_{\text{loc}}(\mu, x), \overline{\dim}_{\text{loc}}(\mu, x) \in [\alpha_{\min}, \alpha_{\max}]$ for any $x \in E$.

Lemma 4.8. *In the setting of Theorem 4.6, let $f(\alpha) = \min_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu)\}$ for $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. If $\varepsilon \geq 0$ and $E_{\alpha,\varepsilon} = \{x \in E : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \leq \alpha + \varepsilon\}$, then*

$$f(\alpha) - c\varepsilon \leq \dim_H(E_{\alpha,\varepsilon}) \leq \dim_p(E_{\alpha,\varepsilon}) \leq f(\alpha) + c\varepsilon \quad (4.13)$$

for a constant $c < \infty$ independent of ε .

Proof. The proof is similar to the proof of [5, Proposition 11.4]. We give some details for the convenience of the reader. Let $r = (r_1, \dots, r_m)$ and $p = (p_1, \dots, p_m)$ be the constant values of the mappings r and p . Denote moreover, $r_i = \prod_{j=1}^n r_{i_j}$ when $i \in \Sigma_n$. We set $\tau = \tau_q(\mu)$ and choose $q \in \mathbb{R}$ so that $f(\alpha) = q\alpha - \tau$. Then we define a probability measure ν on X with $\text{spt}(\nu) = E$ by setting

$$\nu(E_{ii}) = p_i^q r_i^{-\tau} \nu(E_i) = \mu_i^q r_i^{-\tau} = \mu(E_i)^q r_i^{-\tau} \quad (4.14)$$

for $i \in \Sigma_*$ and $i \in \{1, \dots, m\}$. Recall that $\sum_{i=1}^m p_i^q r_i^{-\tau} = 1$ by Theorem 4.2 and Lemma 4.4.

For all $i \in \Sigma$ the condition (M8) implies that

$$\underline{\dim}_{\text{loc}}(\nu, x_i) = \liminf_{n \rightarrow \infty} \log \nu(E_{i|n}) / \log \text{diam}(E_{i|n}), \quad (4.15)$$

$$\overline{\dim}_{\text{loc}}(\nu, x_i) = \limsup_{n \rightarrow \infty} \log \nu(E_{i|n}) / \log \text{diam}(E_{i|n}), \quad (4.16)$$

and similar formulas apply for μ .

Let $\eta > 0$. Using (4.14) and (M6), we find $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ so that if $0 < \delta \leq \delta_0$ and $n \geq n_0$, then

$$\begin{aligned} \nu(\{x_i \in E : \mu(E_{i|n}) < \text{diam}(E_{i|n})^{\alpha+\varepsilon+\eta}\}) \\ = \nu(\{x_i \in E : \mu(E_{i|n})^{-\delta} \text{diam}(E_{i|n})^{\delta(\alpha+\varepsilon+\eta)} \geq 1\}) \\ \leq \sum_{i \in \Sigma_n} \mu(E_i)^{-\delta} \text{diam}(E_i)^{\delta(\alpha+\varepsilon+\eta)} \nu(E_i) \leq \sum_{i \in \Sigma_n} \mu_i^{q-\delta} r_i^{\delta(\alpha+\varepsilon+\eta/2)-\tau} \leq \gamma^n, \end{aligned}$$

where $\gamma < 1$ is independent of n . For the last estimate, see [5, Lemma 11.3]. In the second to last estimate we used (M6) to conclude that $\text{diam}(E_i)^{\delta(\alpha+\varepsilon+\eta)} < r_i^{\delta(\alpha+\varepsilon+\eta/2)}$ for all $i \in \Sigma_n$. Summing the above estimate over all $n \geq n_0$, and letting $\delta \downarrow 0$, this implies that

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq \alpha + \varepsilon \quad (4.17)$$

for ν -almost all $x \in X$. A similar calculation gives

$$\underline{\dim}_{\text{loc}}(\mu, x) \geq \alpha - \varepsilon \quad (4.18)$$

for ν -almost all x . Thus, in particular, we have

$$\nu(X \setminus E_{\alpha,\varepsilon}) = 0. \quad (4.19)$$

From (4.14), it follows that

$$\frac{\log \nu(E_i)}{\log \text{diam}(E_i)} = q \frac{\log \mu(E_i)}{\log \text{diam}(E_i)} - \tau \frac{\log r_i}{\log \text{diam}(E_i)}$$

for all $i \in \Sigma_n$. Using (M3) and (M6), we observe that $\log r_i / \log \text{diam}(E_i) \rightarrow 1$ as $|i| \rightarrow \infty$. Combined with (4.15)–(4.16) and (4.17)–(4.18), this gives $\overline{\dim}_{\text{loc}}(\nu, x) \leq q\alpha + |q|\varepsilon - \tau = f(\alpha) + |q|\varepsilon$ and similarly $\underline{\dim}_{\text{loc}}(\nu, x) \geq f(\alpha) - |q|\varepsilon$ for all $x \in E_{\alpha,\varepsilon}$. Together with (4.19), these estimates readily imply that $f(\alpha) - |q|\varepsilon \leq \dim_H(E_{\alpha,\varepsilon}) \leq \dim_p(E_{\alpha,\varepsilon}) \leq f(\alpha) + |q|\varepsilon$. \square

To finish the paper, we show how the local L^q -spectrum can be used in the setting of Theorem 4.2. We introduce a coarse type local multifractal formalism for the spectrum

$$f_H(\alpha, x) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \dim_H(\{y \in B(x, r) : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) \leq \alpha + \varepsilon\})$$

for $x \in X$ and $\alpha \geq 0$. The corresponding packing spectrum, $f_p(\alpha, x)$ is defined by replacing \dim_H by \dim_p above. Let $\alpha_{\min}(x) = \min\{\frac{\log p_i(x)}{\log r_i(x)} : i \in \{1, \dots, m\}\}$ and $\alpha_{\max}(x) = \max\{\frac{\log p_i(x)}{\log r_i(x)} : i \in \{1, \dots, m\}\}$ be the asymptotic derivatives of $q \mapsto \tau_q(\mu, x)$.

Theorem 4.9. Let $\{E_i : i \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M8). If $p_{i|n} \rightarrow p(x_i)$ uniformly for all $i \in \Sigma$, then

$$f_H(\alpha, x) = f_p(\alpha, x) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau_q(\mu, x)\}$$

for all $x \in E$ and $\alpha_{\min}(x) \leq \alpha \leq \alpha_{\max}(x)$.

Proof. Let $x \in E$, $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$, $p(x) = (p_i(x))_{i=1}^m$ and suppose that ν is the probability measure on X defined using the weights $p(x)$, that is, $\nu(E_{ii}) = p_i(x)\nu(E_i) = P_i(x)$ for each $i \in \Sigma_*$ and $i \in \{1, \dots, m\}$.

Let $\varepsilon > 0$ and $c > 1$. Then, as $p_i \rightarrow p(x_i)$ uniformly and $y \mapsto p(y)$ is continuous, there is $r_0 > 0$ so that

$$p_i^i/c \leq p_i(x) \leq cp_i^i$$

whenever $E_i \subset B(x, r_0)$. This gives

$$C_2 c^{-|i|} \mu(E_i) \leq \nu(E_i) \leq C_3 c^{|i|} \mu(E_i)$$

with some constants C_2 and C_3 and, consequently,

$$\frac{\log C_3 + |i| \log c}{\log \text{diam}(E_i)} + \frac{\log \mu(E_i)}{\log \text{diam}(E_i)} \leq \frac{\log \nu(E_i)}{\log \text{diam}(E_i)} \leq \frac{\log C_2 - |i| \log c}{\log \text{diam}(E_i)} + \frac{\log \mu(E_i)}{\log \text{diam}(E_i)}.$$

As c can be chosen arbitrarily close to 1, and as $\text{diam}(E_i) \leq \gamma^{|i|}$ for some $\gamma < 1$ (use (M6) and the fact that $\max\{r_i(x) : x \in E \text{ and } i \in \{1, \dots, m\}\} < 1$), this implies that for small $r > 0$

$$\underline{\dim}_{\text{loc}}(\mu, y) - \varepsilon/2 \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) + \varepsilon/2 \quad (4.20)$$

when $y \in B(x, r)$. Observe that (4.15) and (4.16) hold for any measure whose support is contained in E by (M8). Denote $E_{\alpha, \varepsilon, r} = \{y \in B(x, r) : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) \leq \alpha + \varepsilon\}$, $E_0 = \{y \in B(x, r) : \alpha - \varepsilon/2 \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \alpha + \varepsilon/2\}$, and $E_1 = \{y \in B(x, r) : \alpha - \frac{3}{2}\varepsilon \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \alpha + \frac{3}{2}\varepsilon\}$. Now it follows from (4.20) that $E_0 \subset E_{\alpha, \varepsilon, r} \subset E_1$. Combined with Lemma 4.8, this yields

$$f(\alpha) - c\varepsilon \leq \dim_H(E_0) \leq \dim_H(E_{\alpha, \varepsilon, r}) \leq \dim_p(E_{\alpha, \varepsilon, r}) \leq \dim_p(E_1) \leq f(\alpha) + c\varepsilon,$$

where $f(\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\nu)\} = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu, x)\}$ and the constant $c < \infty$ is independent of ε . The claim now follows by letting $r \downarrow 0$ and then $\varepsilon \downarrow 0$. \square

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